Lagrangian Coherent Structures in the Restricted Three-Body Problem

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Apply LCS in the Three-Body Problem

- The restricted three-body model
  - Relatively complex → displays ordered/chaotic behavior
  - Many analysis methods require focused processes
- Lagrangian Coherent Structures (LCS)
  - Relatively new, popular theory (est. Haller, 2000)
  - Limited emphasis in astrodynamics (Gawlik et al., 2009)
  - Chaoticity indicators (Lara et al., 2007; Villac, 2008; Villac and Broshcart, 2009)
  - Give broader view that reveals *additional* insight

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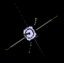
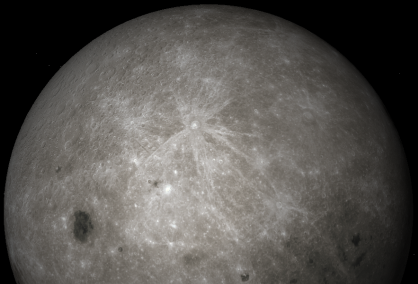
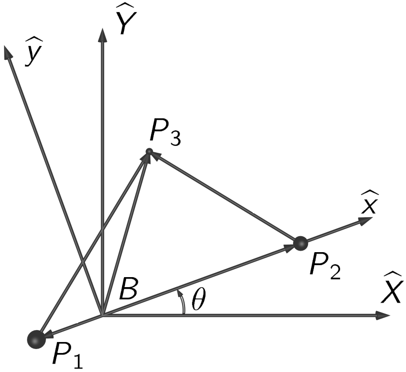


General Formulations and the Restricted Three-Body Problem



Model



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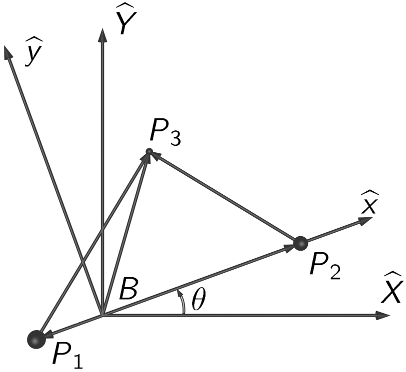
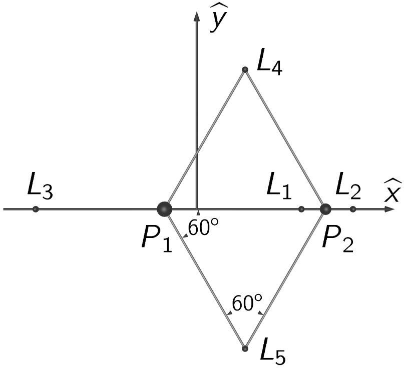
Model



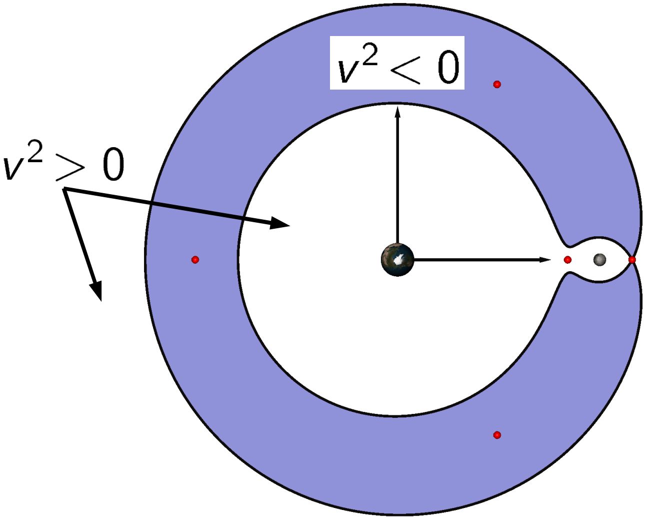
$\hat{Y}$
$\hat{y}$
P_3
$\hat{x}$
P_2
B
θ
$\hat{X}$
P_1



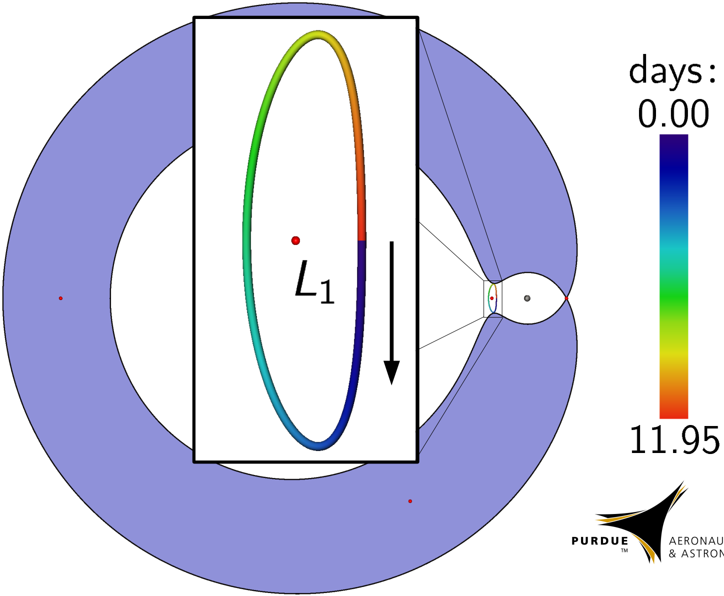
$\hat{y}$
L_4
L_3
L_1
L_2
$\hat{x}$
P_1
60°
P_2
60°
L_5

Boundaries on Motion

- Jacobi integral: $C = 2U^* - v^2$
- Zero Velocity Curves: $v^2 = 0$



A Periodic Orbit



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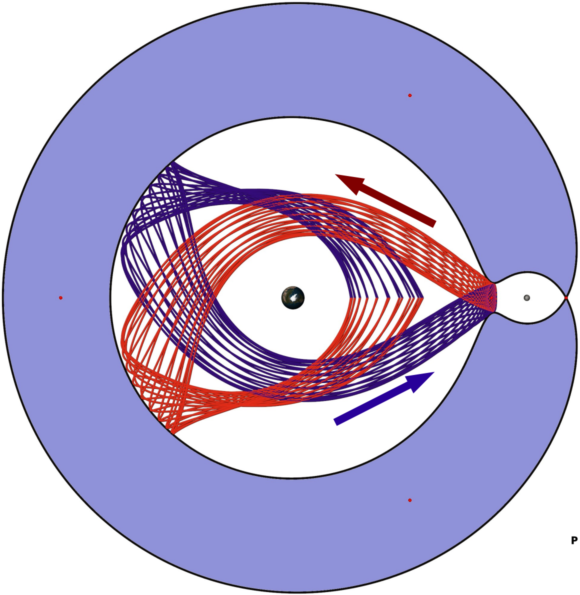
Manifolds Associated with a Periodic Orbit



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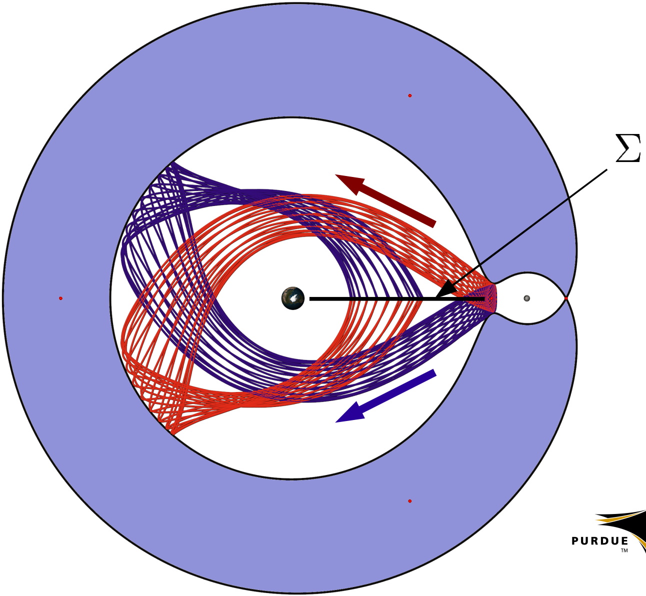
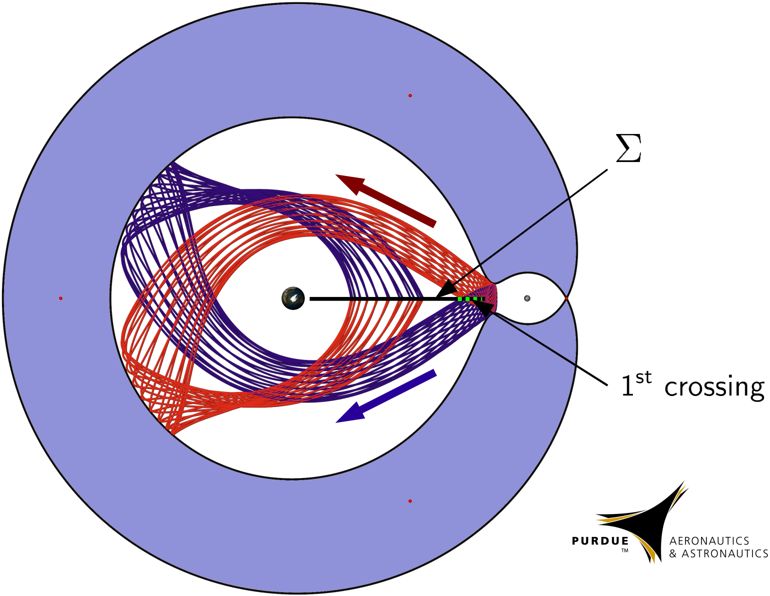
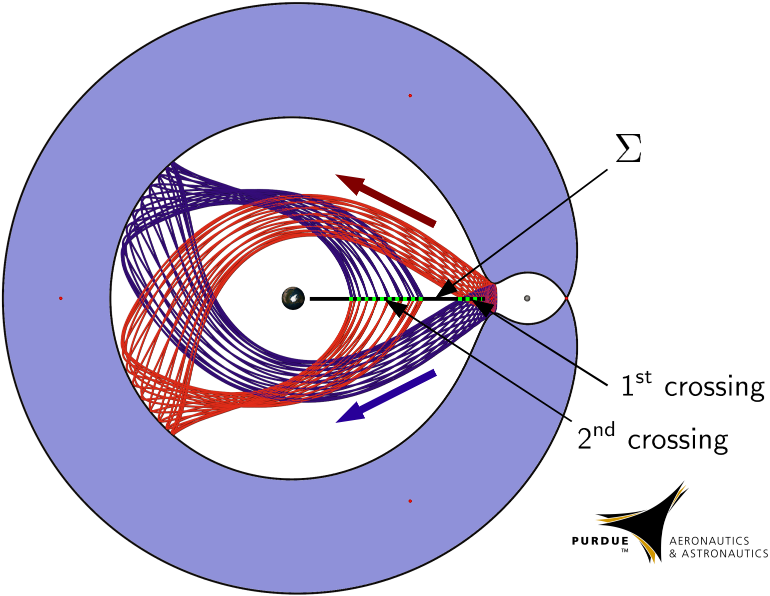
Generating a Manifold Poincaré Section



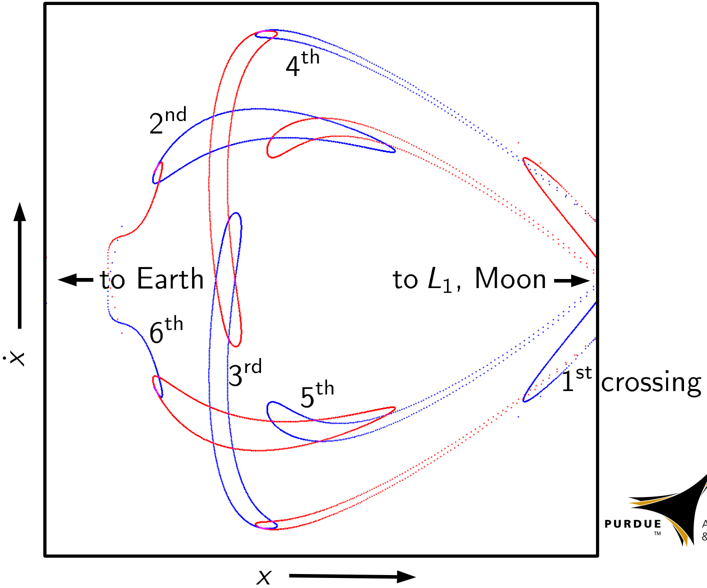
Generating a Manifold Poincaré Section



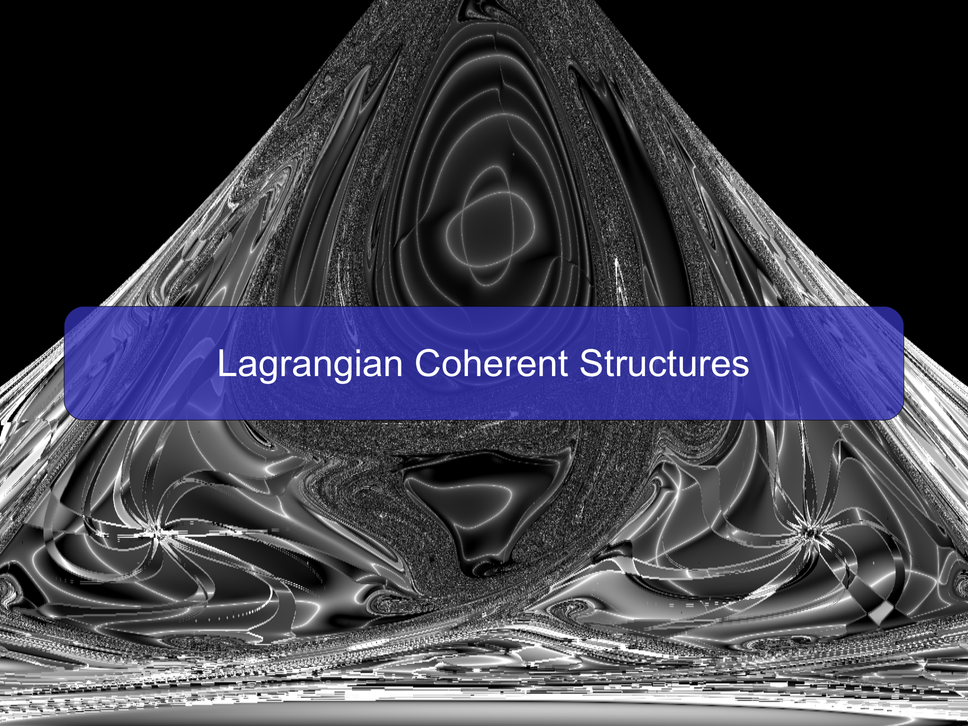
Generating a Manifold Poincaré Section



Manifold Poincaré Section

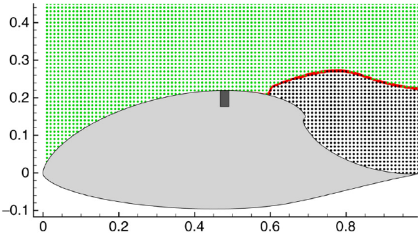


Lagrangian Coherent Structures

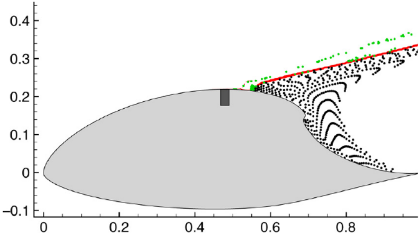


Lagrangian Coherent Structures

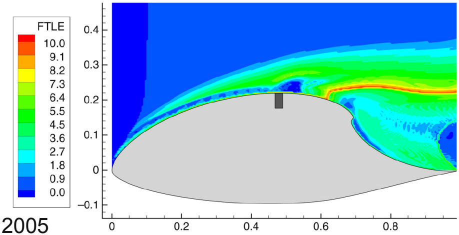
- Introduced ∼10 years ago (G. Haller, 2000)
- Act as material boundaries (S. Shadden et al., 2005)



(a) $t = 0.0$.



(b) $t = 1.5$.



Images from: Shadden et al., 2005

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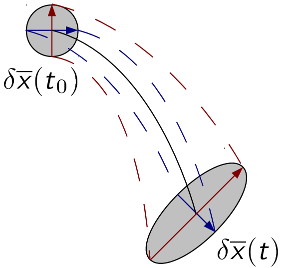
The Finite-Time Lyapunov Exponent (FTLE)

- Lyapunov exponents
  $\Lambda_i = \lim_{t\to\infty} \frac{1}{t} \ln\left|\tilde{\lambda}_i(t)\right| \quad i=1,\ldots,n$
- Describe "stretching"
- Λ_{max}: Characteristic LE

- FTLE – Truncated LE
  $\lambda_i = \frac{1}{|T|} \ln\left|\tilde{\lambda}_i(t)\right| \quad i=1,\ldots,n$
- Same qualitative information
- Generally λ_{max} is *the* FTLE



$\delta\bar{x}(t) = \Phi(t,t_0)\delta\bar{x}(t_0)$

$\Phi(t,t_0) \to \tilde{\lambda}_i(t)$

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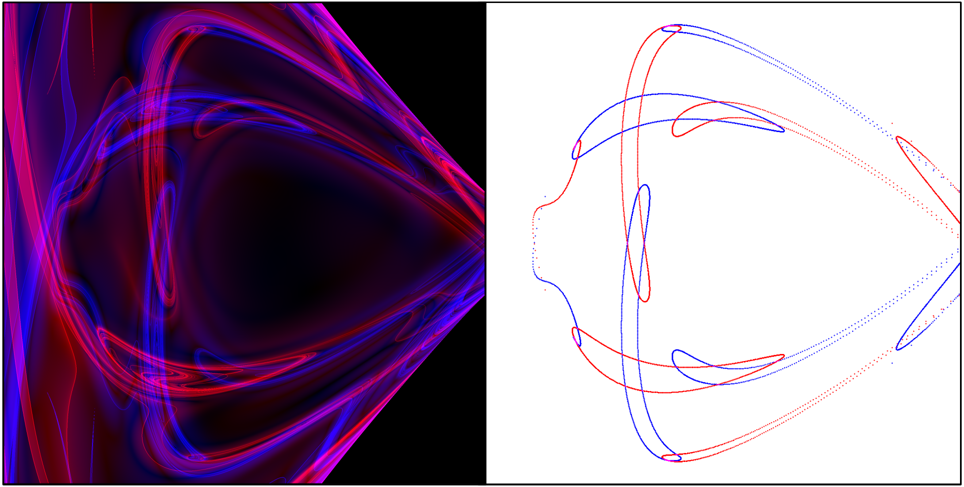
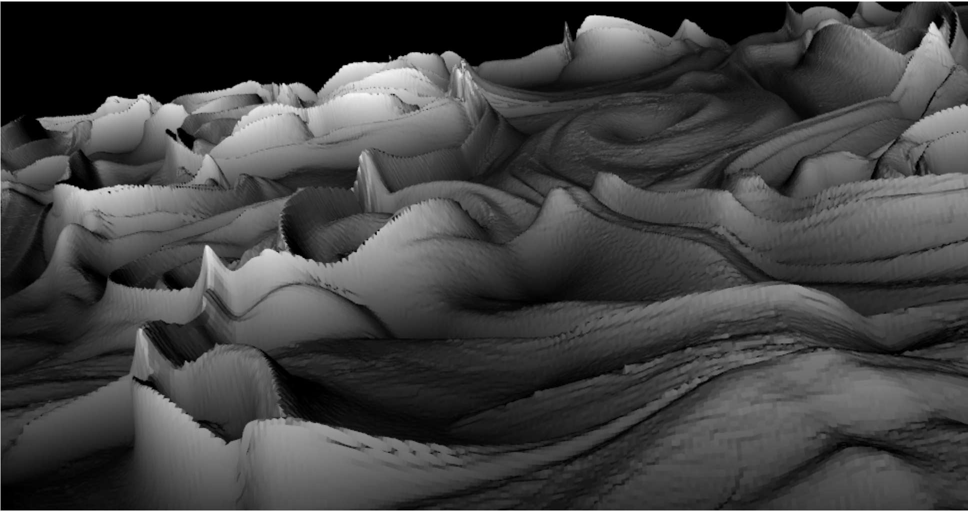
Lagrangian Coherent Structures

- Correlate with invariant manifolds in autonomous systems
- Stable – Forward LCS; Unstable – Reverse LCS

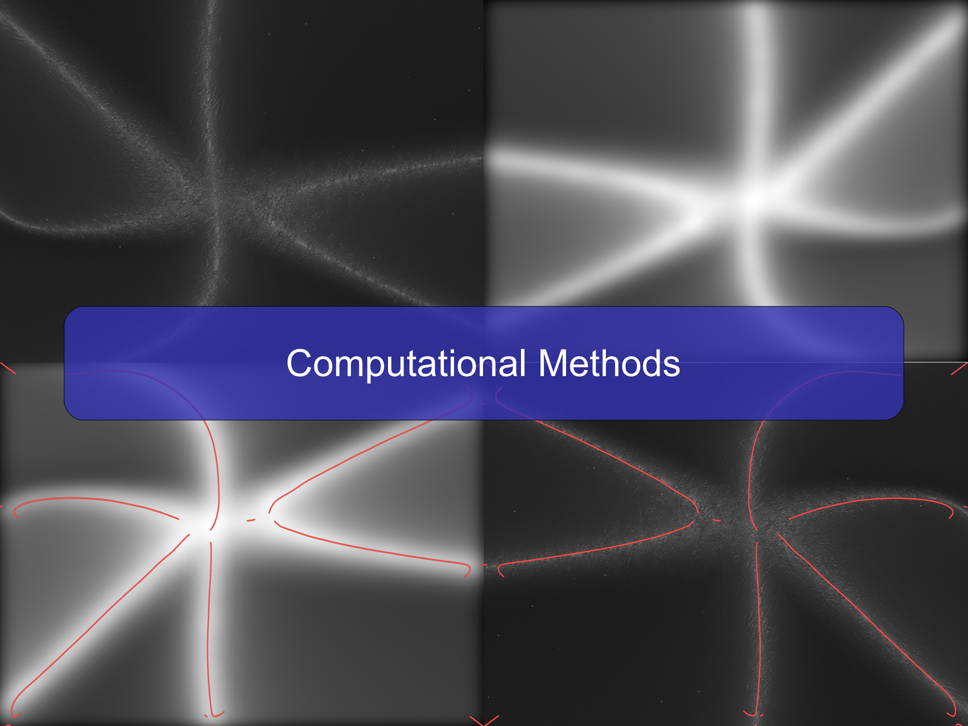


Ridges

- LCS appear as ridges in an FTLE field
- Curves of constrained maxima (i.e., wrt one curvature)



Computational Methods



Computational Methods

- Computing manifold periapse maps
- Parallel computing for FTLE maps
  - Preliminary inspection with Graphics Processing Unit (GPU) parallelization
  - High accuracy maps with CPU parallelization
- Visualization with Teem, Avizo and VTK
  - Specialized interactive visualization tools

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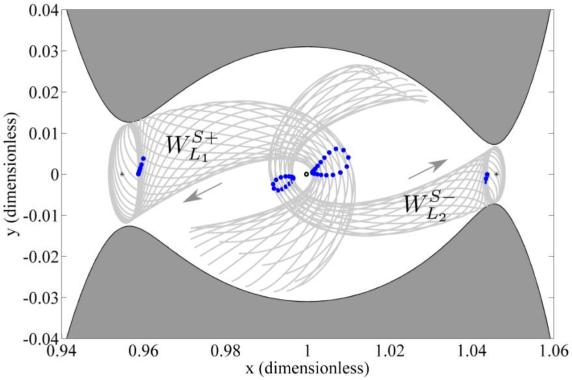
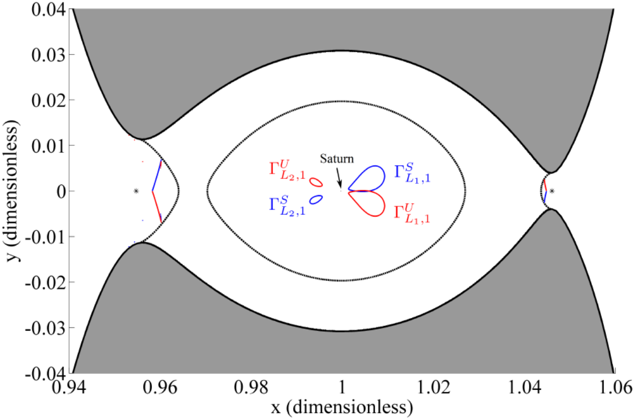
Computing Manifold Periapse Poincaré Maps

$$r = r_p \text{ if } \dot{r} = 0, \ddot{r} \geqslant 0$$



*Source: A. Haapala, 2010 [1]

Computing Manifold Periapse Poincaré Maps



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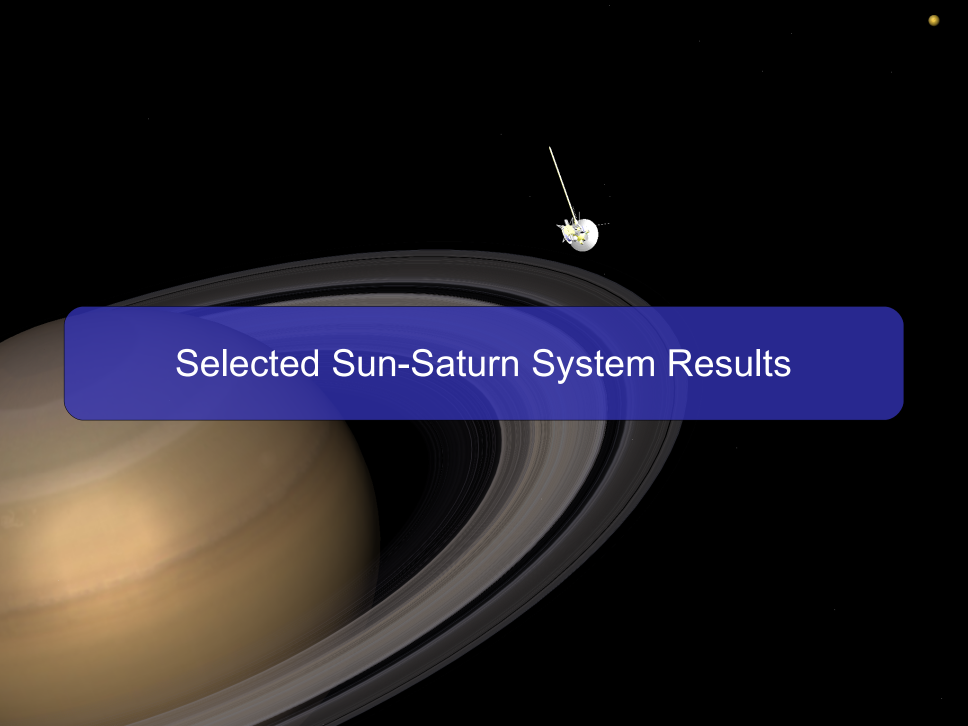
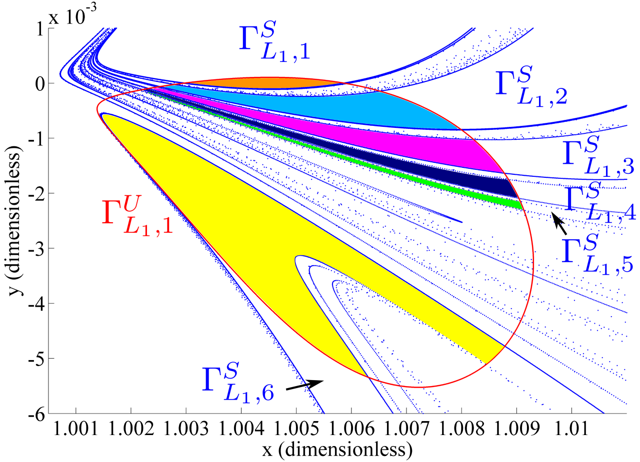


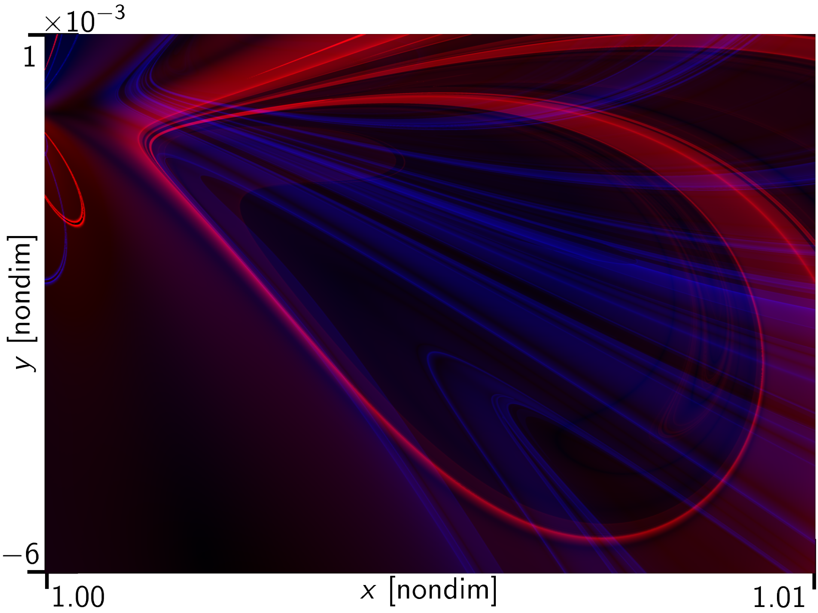
Selected Sun-Saturn System Results

Manifold Periapse Map

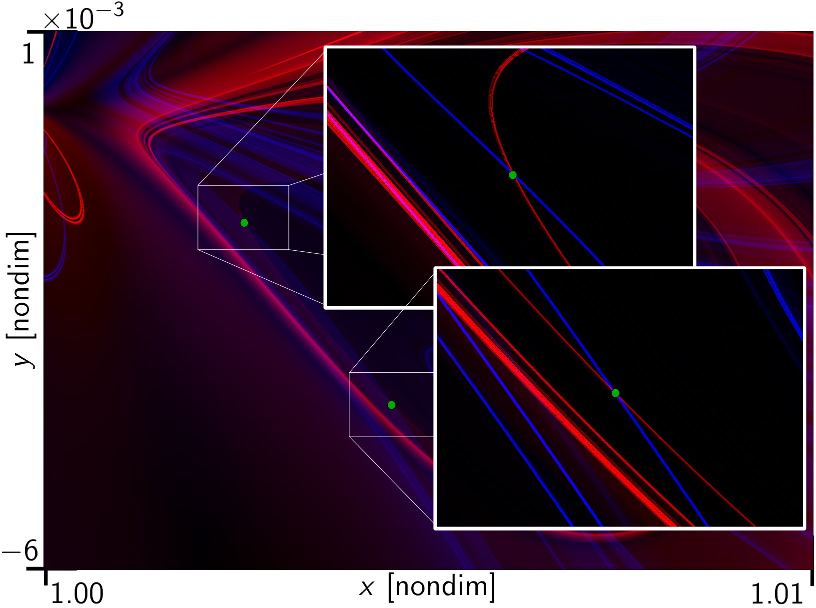


*Source: A. Haapala, 2010 [1]

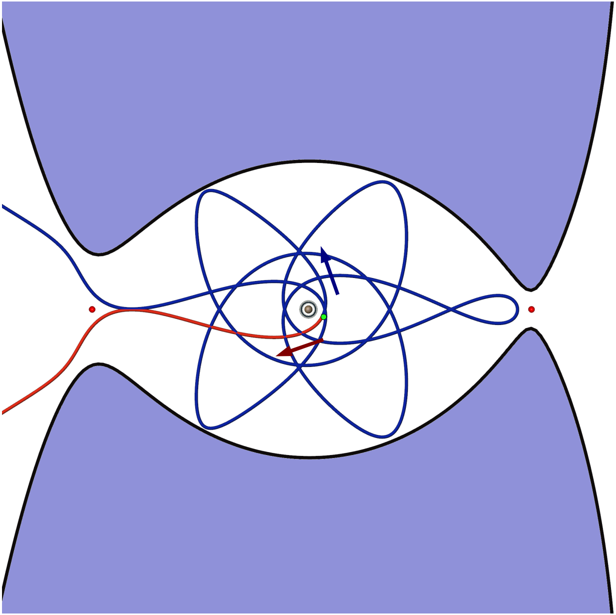
Associated FTLE



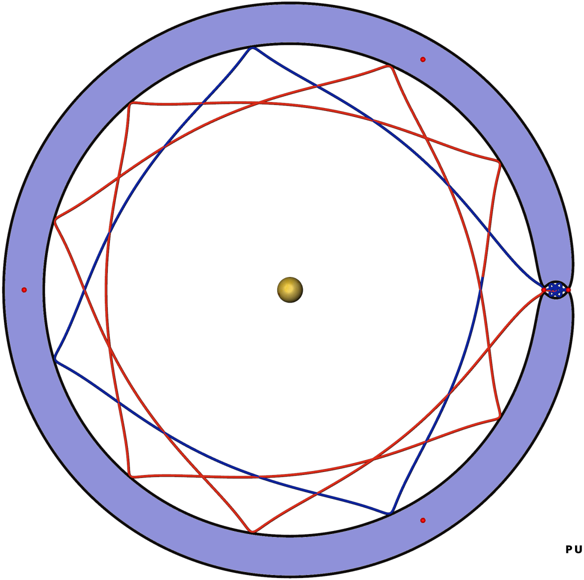
Zoom Focused on Regions of Interest



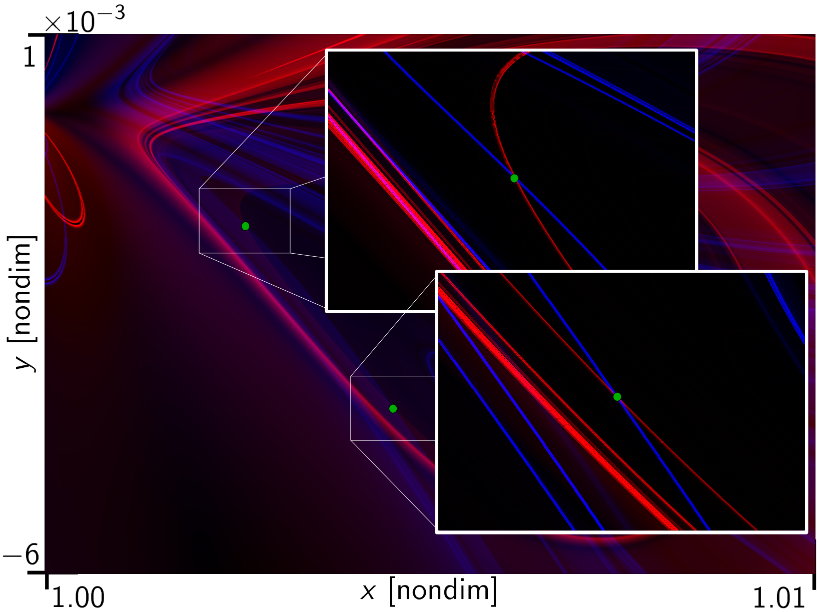
From Top Inset



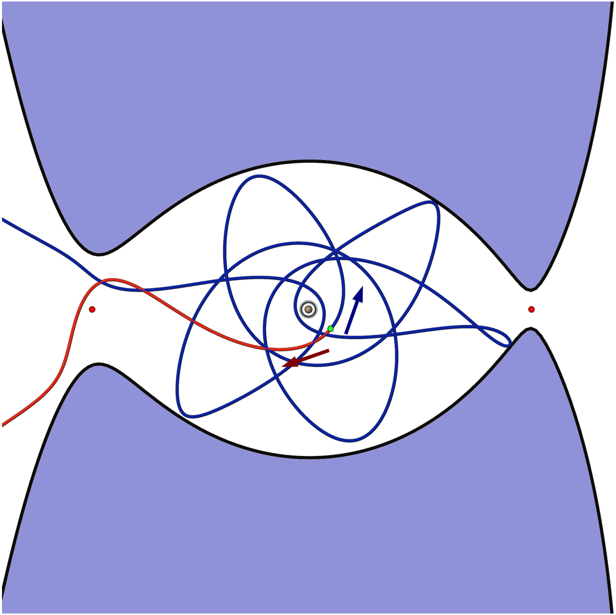
In the P_1 Region



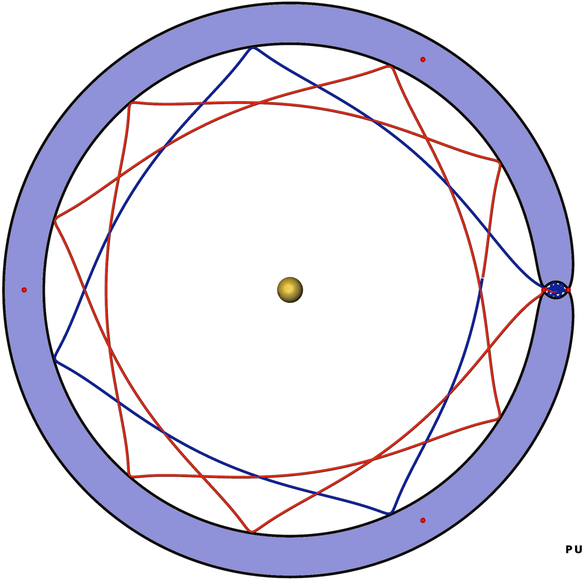
Zoom Focused on Regions of Interest



From Bottom Inset



In the P_1 Region



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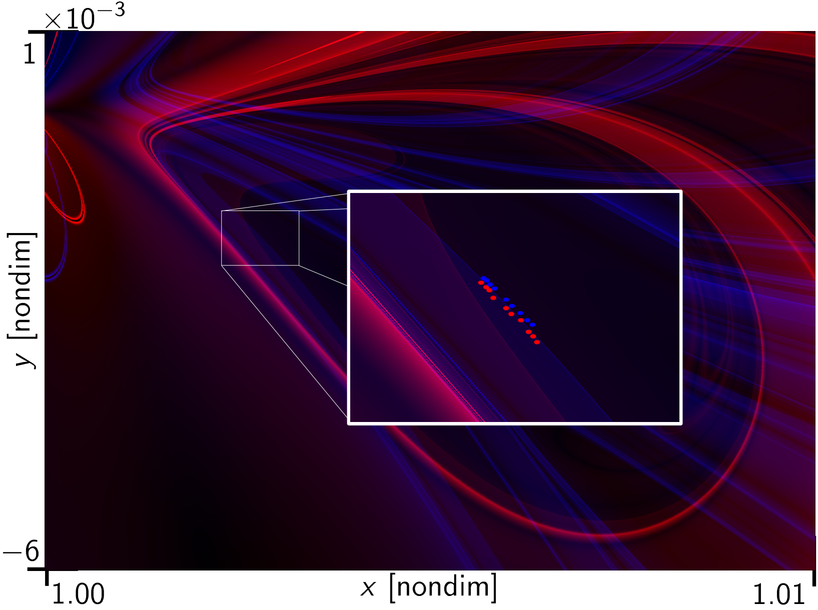
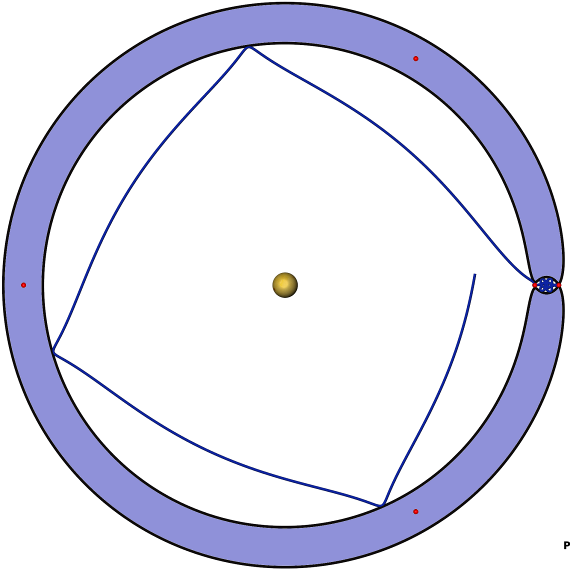
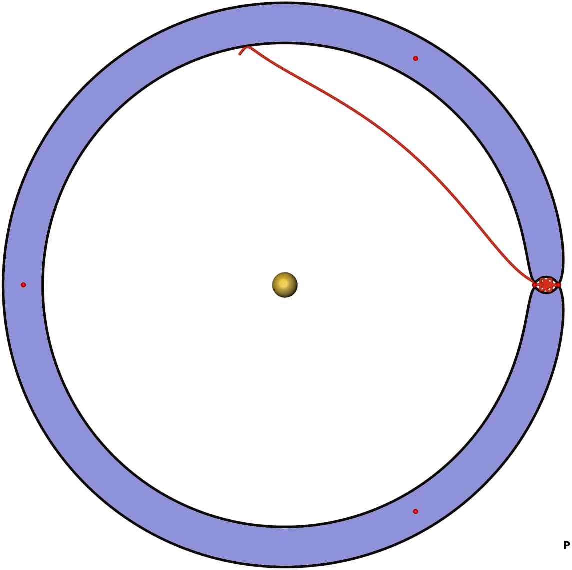
Zoom Along a Particular Ridge



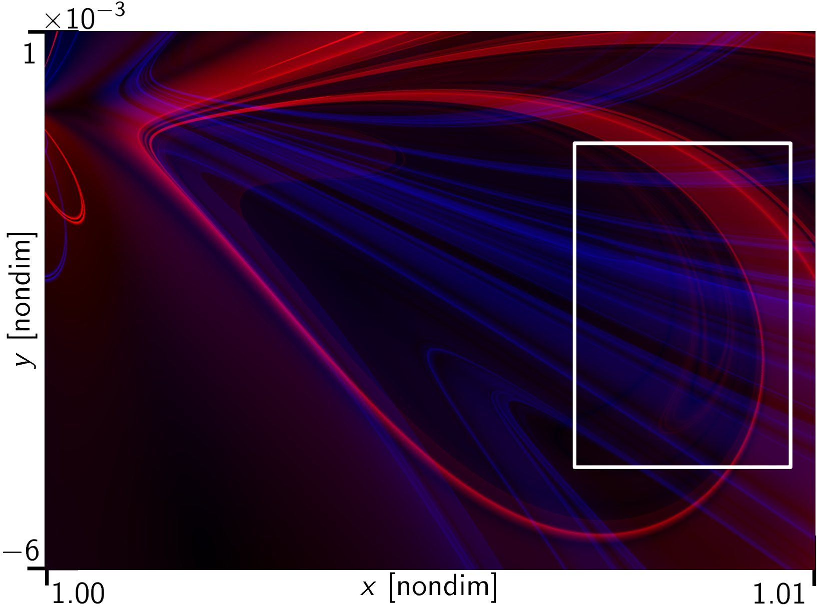
"Over"



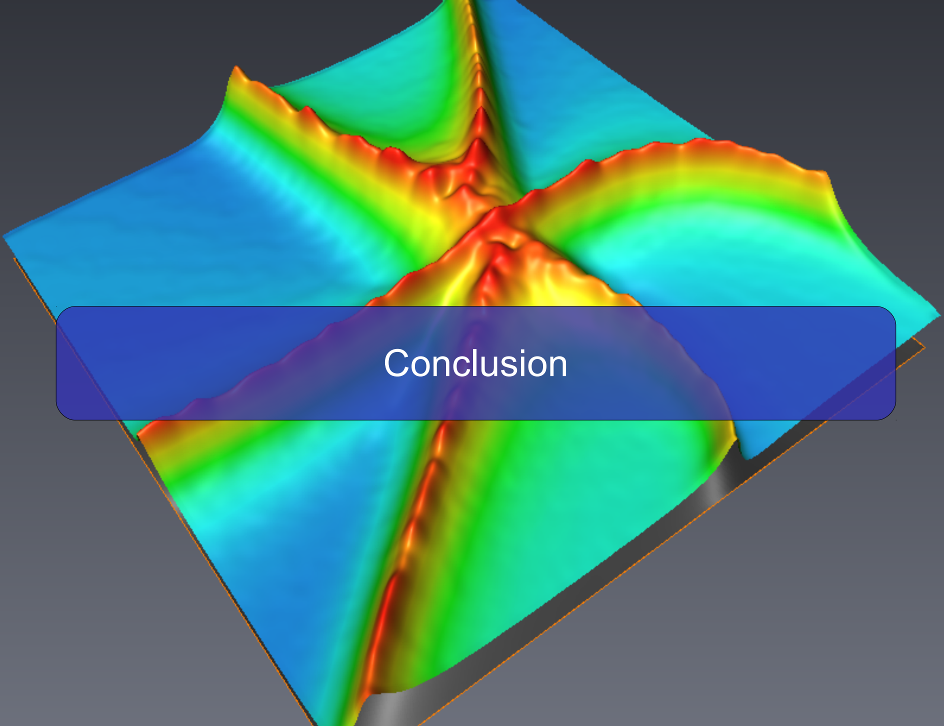
"Under"



Other Interesting Structure



Conclusion



Reflections on LCS

Advantages

- Direct inspection of flow without prior knowledge or intermediate steps
- Straightforward process
- Dense information
- No hyperplane *required*

Disadvantages

- Numerical integration computationally expensive
- Challenging to calculate and extract ridges
- “Structure” resulting from numerical issues

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Conclusions

- Alternative method for identifying known structure
- Identification of previously unknown features in a system
- Indication of qualitatively different regions of motion
- Potentially effective mission analysis and design tool

Future Work

- In planar problems
  - Inspection of equilateral point regions
  - Applications to resonant orbits
  - Planar four-body problem
- In three dimensions
  - FTLE not constrained to hyperplane
  - Investigation in spatial CR3BP
  - Multi-body ephemeris models

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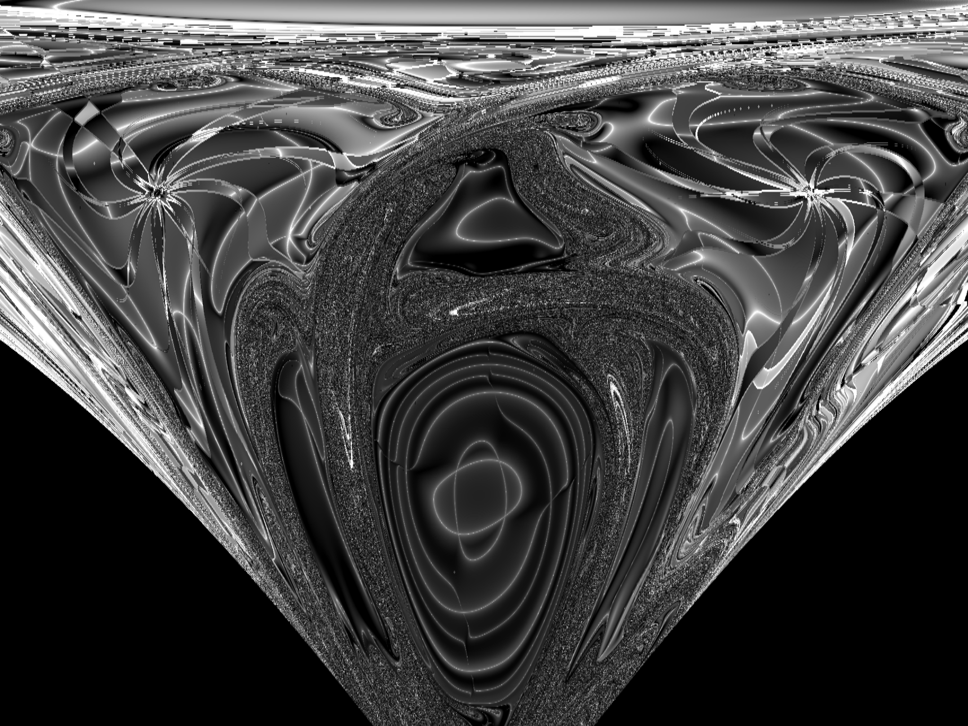

Backup Slides

References

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- D. C. Davis and K. C. Howell, “Trajectory Evolution in the Multi-Body Problem with Applications in the Saturnian System," in *IAF 61st International Astronautical Congress*, (Prague, Czech Republic), September, 2010.
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- S. C. Shadden, F. Lekien, and J. E. Marsden, “Definition and properties of Lagrangian coherent structures from finite-time Lyapunov exponents in two-dimensional aperiodic flows," *Physica D*, vol. 212, pp. 271-304, 2005.

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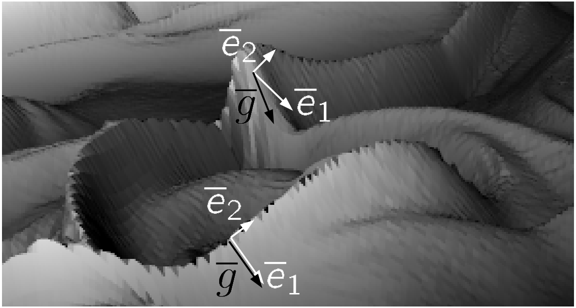
Ridge Detection

- Given a scalar field of FTLE values
- Eigenvectors of Hessian aligned with principle curvatures
- $\overline{e}_1 \triangleq \tilde{\lambda}_1$, maximum curvature; $\overline{e}_2 \triangleq \tilde{\lambda}_2$, minimum curvature
- Gradient points “downhill” – find alignments ($\overline{g} \parallel \overline{e}_1$)



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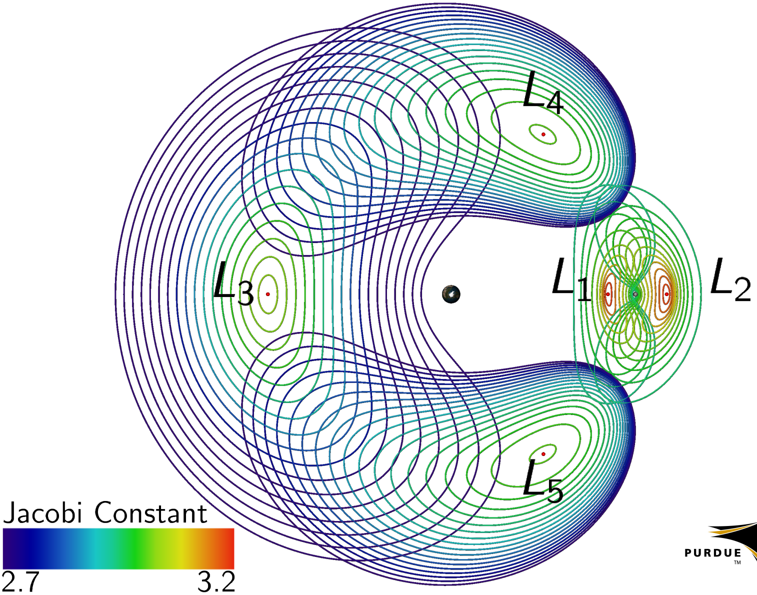
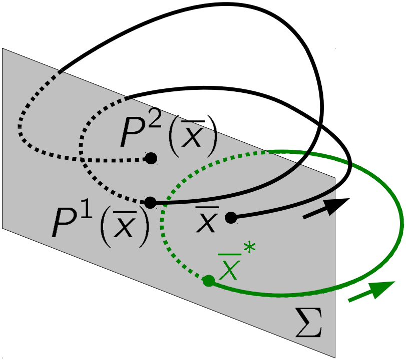
Planar Periodic Orbits



Poincaré Maps



Lagrangian Coherent Structures Defined

- Lagrangian metrics are associated with the evolution of particle characteristics over time
- LCS appear as ridges in an FTLE field (a Lagrangian metric)
- These ridges are sharp, tight structures somewhat described by the connotation of the word *coherent*
- LCS correspond with invariant manifolds in autonomous systems and material boundaries in time-dependent flows

Lagrangian Description of Motion

- Alternate perspective of motion
- Describes characteristics of particles (vs. char. of places)
- Conservation of particle identity
- Label a particle and track it over time
- LCS: Lagrangian metric defined on a Poincaré section

Parallel Processing Schemes

- CPU parallelization
  - OpenMP multi-core implementation
  - 10 cores $\sim 7\times$ speed improvement
  - 24 hours in serial $\rightarrow$ 3.45 hours in parallel
- Graphics Processing Unit (GPU) parallelization
  - Hundreds of processing cores
  - Good for rough initial inspection
  - Interactive speeds

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